Locally observable conditions for the successful implementation of entangling multi-qubit quantum gates

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The information obtained from the operation of a quantum gate on only two complementary sets of input states is sufficient to estimate the quantum process fidelity of the gate. In the case of entangling gates, these conditions can be used to predict the multi qubit entanglement capability from the fidelities of two non-entangling local operations. It is then possible to predict highly non-classical features of the gate such as violations of local realism from the fidelities of two completely classical input-output relations, without generating any actual entanglement.

I. INTRODUCTION

Quantum gates are designed to perform tasks that are far more complex than those achieved by classical gates. To test experimental realizations of quantum gates, it is therefore necessary to identify the characteristic non-classical features that define the quantum nature of the gate. In recent reports of experimental gates, these non-classical features were either characterized by the capability of generating entanglement [1, 2, 3, 4] or by full quantum process tomography [5, 6, 7, 8]. However, the complexity of such tests will increase rapidly as the number of qubits increases. A more intuitive and simple method for testing quantum gate performances is therefore desirable.

In the following we show that the quantum coherent properties of a gate operation can be tested efficiently by using a pair of operations on two complementary sets of orthogonal input states [9, 10, 11]. The classically defined fidelities of these two operations then provide upper and lower bounds for the quantum process fidelity. This estimate of the process fidelity can then be used to predict the performance of the gate for other input states. In particular, the fidelity of generating maximally entangled outputs from product state inputs can be estimated using only the fidelities of local non-entangling operations. The entanglement capability of multi-qubit gates can thus be evaluated without actually generating any entanglement.

II. NOISY GATE OPERATIONS AND PROCESS FIDELITY

Ideal quantum gates are described by a unitary operation $\hat{U}_{00}$, representing the effects on both the amplitude and the phase of an arbitrary quantum state input. Using the computational basis $\{\mid n_z\rangle\}$ for the input states, this ideal operation can be defined by

$$\hat{U}_{00} \mid n_z\rangle = \mid f_n\rangle. \tag{1}$$

However, any experimental implementation will include noise effects due to decoherence. To classify these noise effects, it is useful to expand them into a set of orthogonal unitary operations.

For N qubits, this is easily accomplished by applying all possible combinations of phase and amplitude errors to the input states. Specifically, a phase error is represented by the Pauli matrix Z and an amplitude error is represented by the Pauli matrix X. The set of local operations applied to the input are then given by the single qubit-operations I, X, Z, and $ZX = iY$. The multi-qubit error can therefore be represented by a product of phase and amplitude errors, $\hat{\Pi}_{ij} = \hat{\Phi}_i \hat{A}_j$, where $\hat{\Phi}_i$ is a product operator of N local I and Z operators describing the distribution of phase errors and $\hat{A}_j$ is an product operator of N local I and X operators describing the distribution of amplitude errors. The indices i and j can be given by N-bit binary numbers, so that the location of the errors are given by the digits with a value of 1. With this definition of error operators, a set of 4^N orthogonal operations representing the possible errors of $\hat{U}_{00}$ can be constructed by

$$\begin{aligned} \hat{U}_{ij} &= \hat{U}_{00}\hat{\Pi}_{ij}, \\ \text{with} \quad & \mathrm{Tr}\{\hat{U}_{ij}^\dagger \hat{U}_{kl}\} = 2^N \delta_{ik}\delta_{jl}. \end{aligned} \tag{2}$$

Any operator can now be expressed as a linear combination of the unitary operations $\hat{U}_{ij}$.

In general, a noisy experimental process is described by a superoperator $E(\hat{\rho})$ that transforms an arbitrary input density matrix $\hat{\rho}_{\text{in}}$ into the corresponding output density matrix $\hat{\rho}_{\text{out}} = E(\hat{\rho}_{\text{in}})$. This superoperator can be written as a process matrix with matrix elements $\chi_{ij,kl}$ by expanding it in terms of the orthogonal unitary operations $\hat{U}_{ij}$,

$$E(\hat{\rho}) = \sum_{i,j,k,l} \chi_{ij,kl} \hat{U}_{ij} \hat{\rho} \hat{U}_{kl}^\dagger. \tag{3}$$

The process fidelity is then defined as the overlap of the process matrix with the ideal process $\hat{U}_{00}$, given by the matrix element $\chi_{00,00} = F_{\text{process}}$, and the diagonal elements $\chi_{ij,ij}$ of the process matrix can be interpreted as the probabilities of the phase and amplitude errors ij.

III. OBSERVABLE EFFECTS OF QUANTUM ERRORS

The classical operation performed by a quantum gate is tested by preparing input states in the computational basis states $| n_z\rangle$ and measuring the fidelities of the predicted output states $| f_n\rangle$. The classical fidelity of this operation is equal to the average probability of obtaining the predicted results for all 2^N possible inputs,

$$\begin{aligned} F_z &= \frac{1}{2^N}\sum_n p(f_n|n_z) \\ &= \frac{1}{2^N}\sum_n \langle f_n \mid E(| n_z\rangle\langle n_z |) \mid f_n\rangle. \end{aligned} \tag{4}$$

Using the definition of the output state given in eq. (1), the definition of the errors in eq. (2), and the definition of the process matrix in eq. (3), it can be shown that the classical fidelity is given by

$$F_z = \sum_i \chi_{i0,i0}. \tag{5}$$

The classical fidelity F_z can therefore be identified with the sum of the 2^N diagonal elements of the process matrix corresponding to phase errors only.

Eq.(5) is a quantitative expression of the fact that the classical fidelity F_z is not sensitive to phase errors in the Z-basis inputs. In order to obtain information about these phase errors, it is therefore necessary to use a different input basis. As the operator representation of the errors indicate, it is convenient to use the eigenstates of the local X operators for this purpose. This set of states $\{| n_x\rangle\}$ forms an orthogonal basis set which is complementary to the computational basis in the sense that the overlap between any two states taken from the different basis sets is

$$|\langle n_x \mid n'_z\rangle|^2 = \left(\frac{1}{2}\right)^N. \tag{6}$$

The effect of the unitary operation $\hat{U}_{00}$ on the complementary basis states is given by

$$\hat{U}_{00} \mid n_x\rangle = \mid g_n\rangle. \tag{7}$$

Since the input states $| n_x\rangle$ are eigenstates of the amplitude errors $\hat{A}_j$ this operation is not sensitive to the amplitude errors. The classical fidelity F_x of the complementary operation is therefore given by the sum of the 2^N diagonal elements of the process matrix corresponding to amplitude errors only,

$$F_x = \frac{1}{2^N}\sum_n p(g_n|n_x) = \sum_j \chi_{0j,0j}. \tag{8}$$

By combining the two complementary classical fidelities, it is therefore possible to obtain a fidelity measure that is sensitive to all error syndromes,

$$F_x + F_z - 1 = \chi_{00,00} - \sum_{i,j\neq 0} \chi_{ij,ij}. \tag{9}$$

Since the only positive contribution to this fidelity measure is the process fidelity $\chi_{00,00} = F_{\text{process}}$, this measure is necessarily equal to or lower than the process fidelity. On the other hand, each of the classical fidelities F_z and F_x is necessarily equal to or greater than the process fidelity. Therefore, it is possible to obtain lower and upper bounds of the process fidelity with [9]

$$F_z + F_x - 1 \;\leq\; F_{\text{process}} \;\leq\; \text{Min}\{F_z, F_x\}. \tag{10}$$

This estimate of the process fidelity provides lower limits for all other properties of the quantum gate. It is thus possible to predict the wide range of possible quantum operations from the operations on only 2^{N+1} local input states.

IV. APPLICATION TO ENTANGLING OPERATIONS

One of the essential features of quantum computation is the generation of multi qubit entanglement. The most simple case of N-qubit entanglement generation is a series of controlled-NOT operations, where the Z-state of the first qubit decides whether the other qubits are flipped or not,

$$\hat{U}_{00} = | 0_z\rangle\langle 0_z | \otimes I^{N-1} + | 1_z\rangle\langle 1_z | \otimes X^{N-1}. \tag{11}$$

This operation generates a maximally entangled N-qubit Greenberger-Horne-Zeilinger (GHZ) state if the first qubit input is in an X eigenstate and the remaining $N-1$ qubit inputs are in Z eigenstates,

$$\begin{aligned} \hat{U}_{00} \mid 0_x, 0_z, 0_z, \ldots\rangle &= \frac{1}{\sqrt{2}}(\mid 0_z, 0_z, 0_z, \ldots\rangle \\ &\quad + \mid 1_z, 1_z, 1_z, \ldots\rangle) \\ \hat{U}_{00} \mid 1_x, 0_z, 0_z, \ldots\rangle &= \frac{1}{\sqrt{2}}(\mid 0_z, 0_z, 0_z, \ldots\rangle \\ &\quad - \mid 1_z, 1_z, 1_z, \ldots\rangle) \\ &\vdots \end{aligned} \tag{12}$$

However, if all input qubits are in the Z or X basis, the operation $\hat{U}_{00}$ produces only local Z or X basis output states. The process fidelity F_{process} can therefore be estimated from the local classical fidelities F_x and F_z. Since

the fidelity $F_{\text{ent.}}$ of the entanglement generating operation (12) is necessarily equal to or greater than F_{process}, the minimal fidelity of N-qubit entanglement is equal to $F_z + F_x - 1$.

To obtain a measure of the entanglement capability, it is useful to consider that any state with a GHZ state component of more than 50% ($\langle \text{GHZ} \mid \hat{\rho} \mid \text{GHZ} \rangle > 1/2$) is an N-qubit entangled state. The minimal entanglement capability C can thus be defined as $2F_{\text{ent.}} - 1$, and the estimate obtained from the local classical fidelities F_z and F_x reads

$$C \geq 2F_z + 2F_x - 3. \tag{13}$$

The multi-qubit gate is therefore capable of generating entanglement if the average of the local classical fidelities F_z and F_x is greater than $3/4$. A pair of local non-entangling operations is thus sufficient to demonstrate the entanglement capability of the N-qubit gate.

V. CONDITIONS FOR THE VIOLATION OF LOCAL REALISM

Since the local classical fidelities F_z and F_x define a minimal amount of multi-qubit entanglement capability, it is possible to know whether the gate operation violates the expectations associated with local realism without ever generating actual entanglement.

In the case of a 3 qubit operation, the GHZ paradox is based on the correlation [12]

$$\begin{aligned} K_{\text{GHZ}} &= X \otimes X \otimes X - X \otimes Y \otimes Y \\ &\quad - Y \otimes X \otimes Y - Y \otimes Y \otimes X. \end{aligned} \tag{14}$$

For any combination of classical variables $X, Y = \pm 1$, this correlation is equal to ± 2. However, the quantum mechanical expectation values of the operator K_{GHZ} are ± 4 - twice as high as the result of any local hidden variable theory. Quantum states can therefore violate the GHZ inequality $|K_{\text{GHZ}}| \leq 2$ by as much as a factor of two.

If the entangling operation given by eq. (12) works perfectly, it generates output states with the extremal value of $|K_{\text{GHZ}}| = 4$, clearly violating local realism. If the process fidelity is smaller than one, the minimal value of the above correlation is obtained by assuming that all errors result in the opposite eigenvalue of K_{GHZ},

$$|K_{\text{GHZ}}| \geq 8F_{\text{process}} - 4. \tag{15}$$

Local realism is therefore violated for $F_{\text{process}} > 3/4$. This condition is necessarily fulfilled if the average of the classical fidelities F_z and F_x exceeds $7/8$,

$$\frac{1}{2}(F_z + F_x) > 7/8. \tag{16}$$

An average classical fidelity greater than 7/8 in the two local operations on X and Z basis input qubits therefore ensures that the operation of the three qubit gate can generate a violation of local realism from non-entangled inputs. Similar conditions can be derived for any number of qubits.

VI. CONCLUSIONS

The performance of a quantum gate can be estimated efficiently by testing only two sets of complementary input states, e.g. the computational Z-basis and the X basis generated by local Hadamard gates on Z-basis inputs. By obtaining estimates of the process fidelity, it is possible to predict the performance of the gate for any other set of input states. As a result, the performance of entangling gates can be estimated from the non-entangling local operations observed in the Z and X basis. The quantum features of such multi-qubit gates can thus be predicted from the performance of entirely classical local operations. In particular, it is possible to show that a gate operation violates local realism without actually generating any entangled state.

Acknowledgements

This work was supported in part by Core Research for Evolutional Science and Technology, Japan Science and Technology Agency (JST-CREST).

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